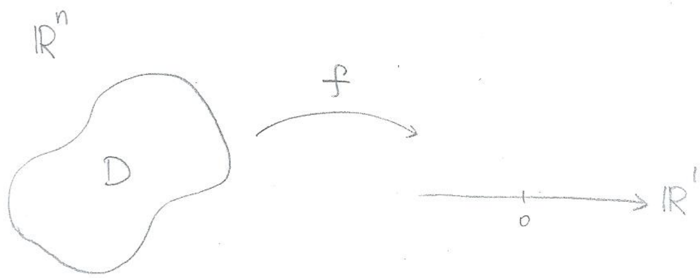
§ Functions of Several Variables :

Let D be a set of n-tuples of real numbers $(x_1 \dots x_n)$, then

$f : D \longrightarrow \mathbb{R}$ is a function of n-variables.



For example, $D = \mathbb{R}^2$, $f(x,y) = x^2 + y^2$

D : domain of f

$R := \{ f(x_1, \dots x_n) \mid (x_1 \dots x_n) \in D \}$ be the range of f.

Terminologies : Let $X = (x_1 \dots x_n)$, $r > 0$

- $B_r(X) = \{ y \in \mathbb{R}^n \mid |X - y| < r \}$

  be the open ball of radius $r > 0$.

- Any subset $S \subset \mathbb{R}^n$, we call

  S a open set iff for any $X \in S$, we can find $r > 0$ such that $B_r(X) \subset S$.

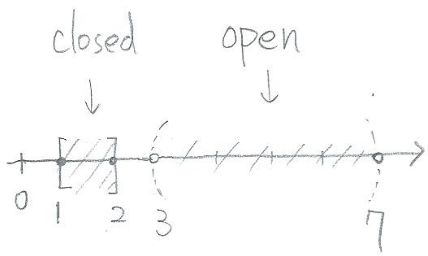
- We call S a closed set iff S^c (complement of S) is open.
- We call S a bounded set iff $S \subset B_r(x)$ for some $x \in \mathbb{R}^n$, $r > 0$.
- S is unbounded if it is not bounded.

<u>Example</u>: In $\mathbb{R}^1$, $[1, 2]$ is closed, $(3, 7)$ is open. See Pic.1

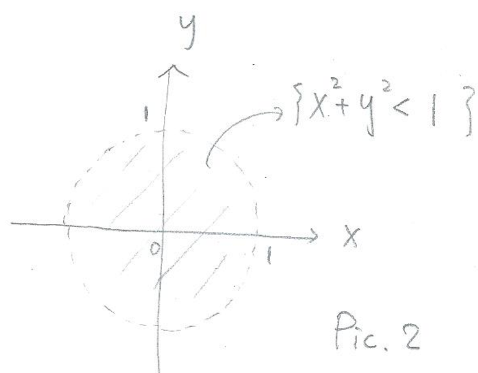
In $\mathbb{R}^2$, $\{x^2+y^2<1\}$ is open

$\{y \geq x^2\}$ is closed

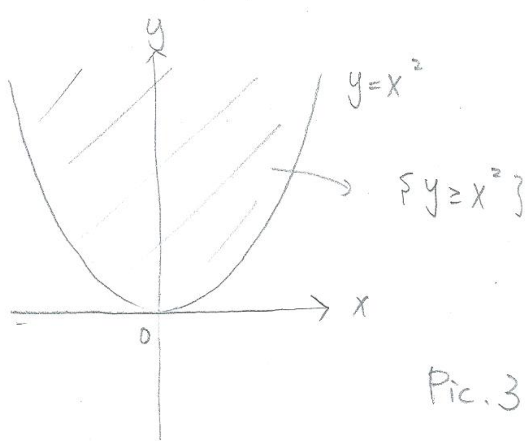
See Pic 2, 3



Pic.1



Pic. 2



Pic.3.

Roughly speaking: Open sets are those set without bdy

Closed sets are those with bdy.

"Q: What's the bdy points?"

Level curves & Level surfaces:

Let $f: \underset{\mathbb{R}^2}{\overset{\cap}{D}} \longrightarrow \mathbb{R}$. Then for any $h \in \mathbb{R}$, we can define

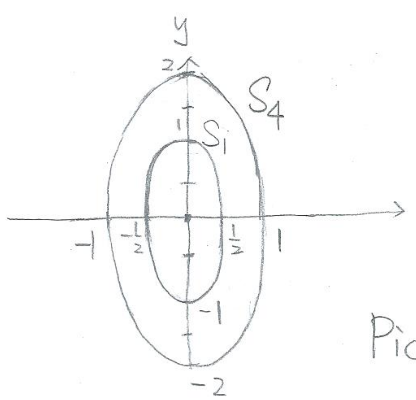
$$S_h := \{ x \in \mathbb{R}^2 \mid f(x) = h \} \quad (\text{or simply write } \{f(x) = h\})$$

When f is smooth, S_h is a curve for "generic" $h \in R$.
(range.
We call S_h the level curve of f.

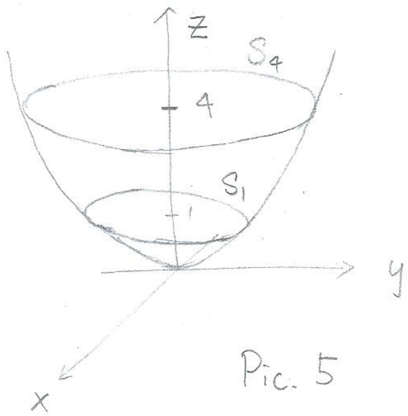
Example: $f(x,y) = 4x^2 + y^2 \quad (R = \{ r \in \mathbb{R} \mid r \geq 0 \})$

When $h = 0$, $S_h = \{0\}$

$h > 0$, $S_h = \{ 4x^2 + y^2 = h \}$ See Pic. 4 and Pic 5.



Pic. 4



Pic. 5

Similarly, we can define the level surfaces as follows

Let $f: \underset{\mathbb{R}^3}{\overset{}{D}} \longrightarrow \mathbb{R}$ (smooth). Then we can define $S_h = \{ x \in D \mid f(x) = h \}$ for any $h \in \mathbb{R}$. S_h will be a surface for generic h.

Example: $f(x,y,z) = x^2 + y^2 + z^2$

$$S_1 = \{ x^2+y^2+z^2 = 1 \} = \text{unit sphere}$$

$$S_4 = \{ x^2+y^2+z^2 = 4 \} = \text{sphere with radius } 2.$$

Limit and continuity in $\mathbb{R}^n$

In $\mathbb{R}^1$, we have

$$\lim_{x \to x_0} f(x) = L$$

iff $|f(x) - L|$ goes to 0 as $|x - x_0|$ goes to 0, $x \neq x_0$,

iff $|f(x) - L| < \varepsilon \longrightarrow 0$ when $0 < |x - x_0| < \delta(\varepsilon)$

for some positive function δ.

In $\mathbb{R}^n$, we have the similar definition:

Let $f : \underset{\mathbb{R}^n}{\underset{\cap}{D}} \longrightarrow \mathbb{R}$. We say $\lim\limits_{x \to x_0} f(x) = L$

iff there exists $\delta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that

$$|f(x) - L| < \varepsilon \quad \text{when} \quad |x - x_0| < \delta(\varepsilon).$$

(Here $x, x_0 \in \mathbb{R}^n$, so $|x - x_0| = \text{dist}(x, x_0)$)

<u>Properties</u>: Let $\lim\limits_{x \to x_0} f(x) = L$ and $\lim\limits_{x \to x_0} g(x) = M$. Then

- $$\lim_{x \to x_0} (f(x) + c\, g(x)) = \lim_{x \to x_0} f(x) + c \lim_{x \to 0} g(x)$$
  $$= L + cM$$

- $$\lim_{x \to x_0} (f(x))^r \cdot (g(x))^s = L^r \cdot M^s$$ for any $r, s \in \mathbb{R}$ and RHS is well-defined.

<u>Def</u>: f is continuous at $x_0 \Leftrightarrow \lim\limits_{x \to x_0} f(x) = f(x_0)$.

<u>Thm</u>: f is a continuous function (f continuous at every point in D) iff $f^{-1}(S)$ is open for any open set S.

Proof: First, suppose " $f^{-1}(S)$ is open for any open set S " is true.

By definition, we want to show that

for any $\varepsilon > 0$, $\exists\, \delta > 0$ such that

$$|\vec{x} - \vec{x}_0| < \delta \Rightarrow |f(\vec{x}) - f(\vec{x}_0)| < \varepsilon .$$

Take $S = B_\varepsilon(f(\vec{x}_0))$. We have

$$f^{-1}(S) = \{ \vec{x} \mid |f(\vec{x}) - f(\vec{x}_0)| < \varepsilon \} .$$

Notice that $\vec{x}_0 \in f^{-1}(S)$, and by our condition $f^{-1}(S)$ is open.

So $\exists\, B_\delta(\vec{x}_0) \subset f^{-1}(S)$, for some $\delta > 0$. This implies

$$|\vec{x} - \vec{x}_0| < \delta \Rightarrow \vec{x} \in B_\delta(\vec{x}_0) \Rightarrow |f(\vec{x}) - f(\vec{x}_0)| < \varepsilon .$$

Conversely, if f is continuous and S is an open set in $\mathbb{R}$, we want to prove that $f^{-1}(S)$ is open.

Let $\vec{x} \in f^{-1}(S)$, $\Rightarrow f(\vec{x}) \in S$. Because S is open,

$\exists\, B_\varepsilon(f(\vec{x})) \subset S$. Since S is continuous, $\exists\, \delta > 0$ such that

$f(B_\delta(\vec{x})) \subset B_\varepsilon(f(\vec{x}))$. So $B_\delta(\vec{x}) \subset f^{-1}(B_\varepsilon(f(\vec{x}))) \subset f^{-1}(S)$.

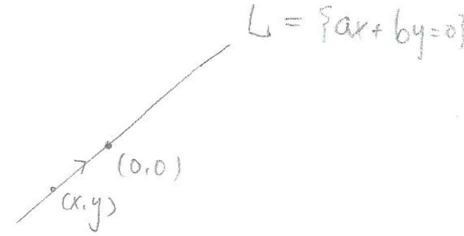
This implies that $f^{-1}(S)$ is an open set.

Two-Path for Nonexistence of limit:

Let $f : \underset{\underset{\mathbb{R}^2}{\cap}}{D} \to \mathbb{R}$. For any line $L = \{ax + by = 0\}$

passing through $(0,0)$, we can consider the limit

$$\lim_{\substack{(x,y)\to(0,0) \\ (x,y)\in L}} f(x,y)$$



Example: $f(x,y) = 2x^2 + xy - y^2$

$$L = \{2x + y = 0\}$$

$$\lim_{\substack{(x,y)\to(0,0) \\ (x,y)\in L}} f(x,y) = \lim_{\substack{(x,y)\to(0,0) \\ 2x+y=0}} (2x^2 + xy - y^2)$$

$$= \lim_{\substack{x\to 0 \\ y=-2x}} (2x^2 + xy - y^2)$$

$$= \lim_{x\to 0} (2x^2 - 2x^2 - 4x^2) = 0$$

Rmk: We can also take the limit of f at any (x_0, y_0) along a line $\{ax + by + c = 0\}$.
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \ni (x_0, y_0)$

Thm: if $\lim\limits_{\substack{(x,y)\to(x_0,y_0) \\ \text{along } L_1}} f(x,y) \neq \lim\limits_{\substack{(x,y)\to(x_0,y_0) \\ \text{along } L_2}} f(x,y)$

Then $f(x,y)$ is not continuous at (x_0,y_0).

- In fact, we can take limit from two different curves. Not necessarily be lines.

Partial Differentiations

Def: If $f : \underset{\mathbb{R}^2}{\overset{}{D}} \to \mathbb{R}$, $D \subset \mathbb{R}^2$, $(x_0,y_0) \in D$, we define

$$\frac{\partial f}{\partial x}(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}$$

and

$$\frac{\partial f}{\partial y}(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0,y_0+h)-f(x_0,y_0)}{h}.$$

(Geometric meaning?)

Properties: Let $f, g : D \to \mathbb{R}$, $(x_0,y_0) \in D$.

- $\frac{\partial}{\partial x}(f+cg) = \frac{\partial f}{\partial x} + c\frac{\partial g}{\partial x}$.
- $\frac{\partial}{\partial x}(fg) = \frac{\partial f}{\partial x} g + f \cdot \frac{\partial g}{\partial x}$
- (Chain Rule). If we have $x(u,v)$ $y(u,v)$

Then

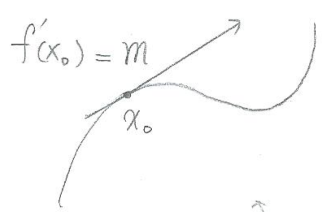
$$\frac{\partial}{\partial u}\Big(f\big(x(u,v), y(u,v)\big)\Big) = \frac{\partial f}{\partial x}\big(x(u,v)\ y(u,v)\big) \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\big(x(u,v)\ y(u,v)\big) \cdot \frac{\partial y}{\partial u}.$$

- $\frac{\partial^2}{\partial x \partial y} f = \frac{\partial^2}{\partial y \partial x} f.$

Differentiability :

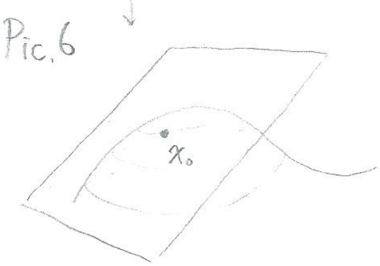
Differentiable $\Leftrightarrow$ $\exists$ linear approximation (tangent , Slop ..)

Pic. 6 .



It's also true for functions with several variables .

Pic.6



$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

$$\Rightarrow \lim_{h \to 0} \frac{f(x_0+h) - \big(f'(x_0)h + f(x_0)\big)}{h} = 0$$

let $L(h) = f'(x_0)h + f(x_0)$: line passing through x_0 .

Def: Let $f: \underset{\mathbb{R}^2}{\underset{\cap}{D}} \longrightarrow \mathbb{R}$. $(x_0, y_0) \in D$. Then we call that

f is differentiable at (x_0, y_0) iff

$$\exists\ L(h, k) = ah + bk + c$$

s.t

$$\lim_{(h,k)\to(0,0)} \frac{f(x_0+h, y_0+k) - ah + bk + c}{\sqrt{h^2+k^2}} = 0$$

(In fact, we have $a = \frac{\partial f}{\partial x}(x_0, y_0)$

$$b = \frac{\partial f}{\partial y}(x_0, y_0)$$

$$c = f(x_0, y_0) \quad)$$

$\{a(x-x_0) + b(y-y_0) + c\}$ = Tangent plane